

ITERATED ARC GRAPHS

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ABSTRACT. The arc graph $\delta(G)$ of a digraph G is the digraph with the set of arcs of G as vertex-set, where the arcs of $\delta(G)$ join consecutive arcs of G . In 1981, Poljak and Rödl characterised the chromatic number of $\delta(G)$ in terms of the chromatic number of G when G is symmetric (i.e., undirected). In contrast, directed graphs with equal chromatic numbers can have arc graphs with distinct chromatic numbers. Even though the arc graph of a symmetric graph is not symmetric, we show that the chromatic number of the iterated arc graph $\delta^k(G)$ still only depends on the chromatic number of G when G is symmetric.

Keywords: Arc graphs, chromatic numbers, free distributive lattices, Dedekind numbers

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1. INTRODUCTION

The arc graph $\delta(G)$ of a digraph G is the digraph with the set $A(G)$ of arcs of G as vertex-set, where the arcs of $\delta(G)$ join consecutive arcs of G . The iterated arc graphs $\delta^k(G)$, $k \geq 1$ are defined recursively by $\delta^k(G) = \delta(\delta^{k-1}(G))$. However it is possible to interpret $\delta^k(G)$ in terms of sequences of vertices of G :

$$\begin{aligned} V(\delta^k(G)) &= \{(u_1, \dots, u_k) \in V(G)^k \mid \\ &\quad (u_i, u_{i+1}) \in A(G) \text{ for } i = 1, \dots, k-1\}, \\ A(\delta^k(G)) &= \{((u_0, \dots, u_{k-1}), (u_1, \dots, u_k)) \mid \\ &\quad (u_0, \dots, u_{k-1}), (u_1, \dots, u_k) \in V(\delta^k(G))\}. \end{aligned}$$

In particular, the iterated arc graphs of complete graphs with loops are the well-known de Bruijn graphs. The iterated arc graphs of transitive tournaments are a folklore construction of graphs with large chromatic numbers and no short odd cycles (see [3]).

We will be investigating chromatic numbers of iterated arc graphs. Here, the chromatic number of a digraph is defined as the minimum

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number of colours needed to colour its vertices so that the endpoints of an arc have different colours. Thus, the direction of an arc has no effect on the chromatic number. In [1], Entriger and Harner give the following relations between the chromatic number of a digraph and that of its arc graph.

Theorem 1 ([1]).

- (i) If $\chi(\delta(G)) \leq n$, then $\chi(G) \leq 2^n$.
- (ii) If $\chi(G) \leq \binom{n}{\lfloor n/2 \rfloor}$, then $\chi(\delta(G)) \leq n$.

Inductively, the bound gives $\theta(\log^{(k)}(\chi(G)))$ behaviour for $\chi(\delta^k(G))$ in terms of $\chi(G)$.

Undirected graphs can be viewed as symmetric digraphs with each edge corresponding to an opposite pair of arcs. In [6], Poljak and Rödl give a characterisation of the chromatic number of the arc graph of an undirected graph.

Theorem 2 ([6]). *If G is an undirected graph, then*

$$\chi(\delta(G)) = \min \left\{ n \mid \chi(G) \leq \binom{n}{\lfloor n/2 \rfloor} \right\}.$$

In particular, if G is undirected, then $\chi(\delta(G))$ depends on $\chi(G)$ alone and not on the structure of G . In contrast, directed graphs with equal chromatic numbers can have arc graphs with distinct chromatic numbers. For instance, let C be the cyclic tournament on three vertices, and T the transitive tournament on three vertices. then $\chi(C) = \chi(T) = 3$, while $\chi(\delta(C)) = 3$ and $\chi(\delta(T)) = 2$.

Now, what about iterated arc graphs of undirected graphs? If G is undirected, then Theorem 2 characterises $\chi(\delta(G))$ in terms of $\chi(G)$, but since $\delta(G)$ is directed, Theorem 2 cannot be used to characterise $\chi(\delta^2(G))$ in terms of $\chi(\delta(G))$.

In this paper, we show that $\chi(\delta^k(G))$ is indeed characterised by $\chi(G)$. The characterisation uses numbers defined in terms of specific posets. Let \overline{K}_n be the antichain of size n . For a poset P , let $\mathcal{I}(P)$ be the poset of ideals (i.e., downsets) of P , ordered by inclusion. Then $\mathcal{I}(P)$ is itself a poset, and the construction can be iterated. We let $b(n, k)$ be the maximum size of an antichain in $\mathcal{I}^k(\overline{K}_n)$ (i.e., the *width* of $\mathcal{I}^k(\overline{K}_n)$). Our main result is the following.

Theorem 3. *For any undirected graph G and any integer $k \geq 1$,*

$$\chi(\delta^k(G)) = \min \{ n \mid \chi(G) \leq b(n, k) \}.$$

In particular, $\mathcal{I}(\overline{K}_n)$ is the boolean lattice with n generators, and $b(n, 1) = \binom{n}{\lfloor n/2 \rfloor}$ by Sperner's theorem. Thus the case $k = 1$ of Theorem 3 is Theorem 2. $\mathcal{I}^2(\overline{K}_n)$ is the free distributive lattice with n generators. The largest known antichain in $\mathcal{I}^2(\overline{K}_n)$ consists of the ideals with exactly 2^{n-1} elements, that is, half the elements of $\mathcal{I}(\overline{K}_n)$. In [5], it has been verified that for n up to 6, this is indeed the unique maximum antichain in $\mathcal{I}^2(\overline{K}_n)$. Thus the known values of $b(n, 2)$ are as follows.

$$b(3, 2) = 4, b(4, 2) = 24, b(5, 2) = 621, b(6, 2) = 492288.$$

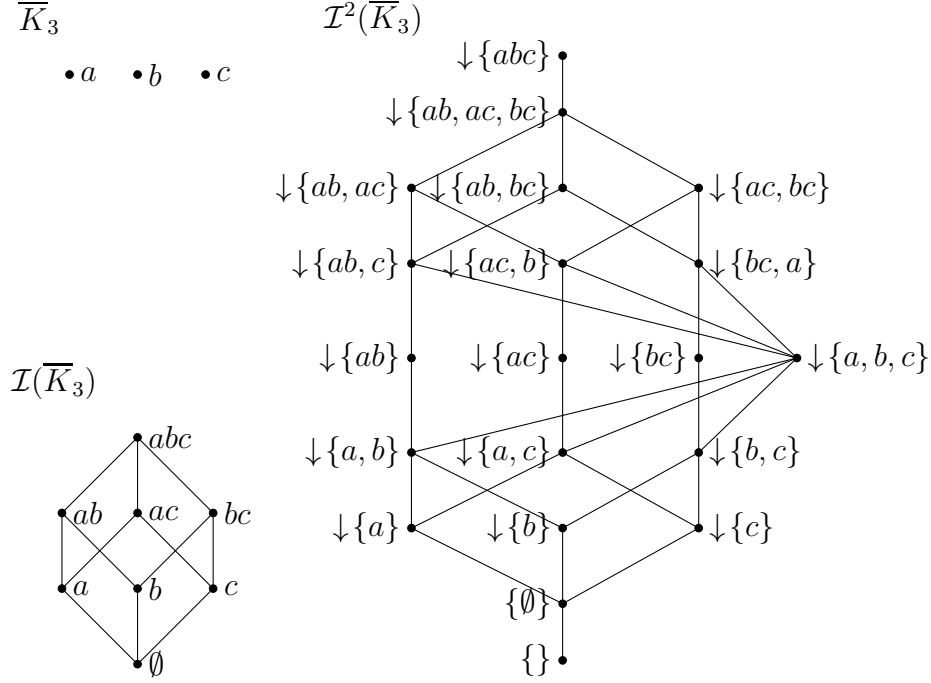
In [5], McHard conjectures that the ideals with exactly half the elements of $\mathcal{I}(\overline{K}_n)$ always constitute the unique maximum antichain of $\mathcal{I}^2(\overline{K}_n)$. Note that the ideals of a given size in $\mathcal{I}^{k-1}(\overline{K}_n)$ constitute a level in $\mathcal{I}^k(\overline{K}_n)$. The posets in which some level constitutes a maximum antichain are called *Sperner posets*. Thus McHard's conjecture implies that the free distributive lattices are Sperner posets. The conjecture seems plausible, though highly difficult. Even the cardinalities of free distributive lattices, known as the Dedekind numbers, are not known in closed form. In [4], it is shown that for any fixed ℓ there exists a polynomial p_ℓ such that $p_\ell(n)$ is the number of elements at the ℓ -th level of $\mathcal{I}^2(\overline{K}_n)$.

By inspecting the 20-element $\mathcal{I}^2(\overline{K}_3)$ —see Figure 1—it is easy to check that $b(3, 3) = 7$. The levels 8, 9, 11, 12 of $\mathcal{I}^3(\overline{K}_3)$ each constitute antichains of size 7, but level 10 is an antichain of size 6. Thus the level of maximum size is not necessarily the middle level. Apart from the trivial cases $b(1, k) = 1$ and $b(2, k) = 2$ for all k , no further values $b(n, k)$ are known.

2. PROOF OF THEOREM 3

We will show that an n -colouring of $\delta^k(G)$ corresponds to a homomorphism from G to a suitably defined digraph $\mathcal{N}(\mathcal{I}^k(\overline{K}_n))$. When G is undirected, its edges must be mapped to the symmetric arcs of $\mathcal{N}(\mathcal{I}^k(\overline{K}_n))$. These symmetric arcs span a graph which retracts to $K_{b(n, k)}$. The details of this argument are provided below.

2.1. The right adjoint of the arc graph construction. Viewed as a digraph functor, δ admits a kind of “right adjoint”. More precisely, there is a construction δ_R such that there exists a homomorphism from $\delta(G)$ to K if and only if there exists a homomorphism from G to $\delta_R(K)$. (Note that our definition of a right adjoint is less restrictive than the

FIGURE 1. $\mathcal{I}^k(\overline{K}_3)$, $k \in \{0, 1, 2\}$.

standard categorial definition, in which a correspondence between morphisms is required.) Here, a *homomorphism* $\phi : G \rightarrow H$ is a map ϕ from the vertex set of G to that of H such that if (u, v) is an arc of G , then $(\phi(u), \phi(v))$ is an arc of H .

For a digraph K , $\delta_R(K)$ is the digraph defined as follows.

- The vertices of $\delta_R(K)$ are the ordered pairs (X, Y) such that X and Y are sets of vertices of K with an arc (x, y) between all vertices x of X and all vertices y of Y .
- The arcs of $\delta_R(K)$ are ordered pairs $((X, Y), (Z, W))$ such that $Y \cap Z \neq \emptyset$.

The sets X, Y of vertices of K used in the definition of $\delta_R(K)$ are allowed to be empty. In particular, if K_0 is the graph with no vertex and no edge, then $\delta_R(K_0)$ is a single vertex, and $\delta_R^2(K_0)$ has three vertices and one arc.

We use the following result.

Lemma 4 ([2]). *Given two digraphs G and K , there exists a homomorphism of $\delta(G)$ to K if and only if there exists a homomorphism of G to $\delta_R(K)$.*

Proof. We include the sketch of an elementary proof to make the paper self contained. First note that a homomorphism $\phi : G \rightarrow H$ induces homomorphisms $\delta(\phi) : \delta(G) \rightarrow \delta(H)$ defined by $\delta(\phi)(u, v) = (\phi(u), \phi(v))$, and $\delta_R(\phi) : \delta_R(G) \rightarrow \delta_R(H)$ defined by $\delta_R(\phi)(X, Y) = (\phi(X), \phi(Y))$. Second, note that there are homomorphisms from $\delta(\delta_R(G))$ to G defined by mapping $((X, Y), (Z, W))$ to any element of $Y \cap Z$, and from G to $\delta_R(\delta(G))$ defined by mapping u to (u^-, u^+) , where u^- and u^+ are respectively the sets of arcs entering and leaving u .

Therefore, if there exists a homomorphism from $\delta(G)$ to K then there exists a homomorphism from $\delta_R(\delta(G))$ to $\delta_R(K)$, which composed with a homomorphism from G to $\delta_R(\delta(G))$ yields a homomorphism from G to $\delta_R(K)$. Similarly, if there exists a homomorphism from G to $\delta_R(K)$, then there exists a homomorphism from $\delta(G)$ to $\delta(\delta_R(K))$, and the latter admits a homomorphism to K . \square

Corollary 5. *Given two digraphs G and K and any $k \geq 1$, there exists a homomorphism of $\delta^k(G)$ to K if and only if there exists a homomorphism of G to $\delta_R^k(K)$.*

A subdigraph H of a digraph G is called a *retract* of G if there exists a homomorphism $\rho : G \rightarrow H$ such that the restriction of ρ to H is the identity.

Lemma 6. *The digraph $\delta_R(K)$ retracts to its subdigraph induced by the vertices (X, Y) such that*

$$Y = \{y \in V(K) \mid (x, y) \in A(K) \text{ for all } x \in X\}$$

and

$$X = \{x \in V(K) \mid (x, y) \in A(K) \text{ for all } y \in Y\}.$$

Proof. Let $\rho : \delta_R(K) \rightarrow \delta_R(K)$ be the map defined by $\rho(X, Y) = (X', Y')$, where X' is the set of common inneighbours of Y , and Y' is the set of common outneighbours of X . Then ρ is easily seen to be a retraction on the prescribed subdigraph. \square

2.2. Nondomination digraphs of posets. The *nondomination digraph* $\mathcal{N}(P)$ of a poset P is the digraph which has the elements of P for vertices, and for arcs the ordered pairs (u, v) such that u is strictly less than v or u and v are incomparable. In other words, if $G = \mathcal{N}(P)$, then $A(G)$ is the complement in $V(G)^2$ of the relation \geq . Note that $K_n = \mathcal{N}(\overline{K}_n)$, where \overline{K}_n is the antichain of size n . The constructions, δ_R , and \mathcal{N} connect to the construction \mathcal{I} of the Introduction as follows.

Lemma 7. *For any poset P , $\delta_R(\mathcal{N}(P))$ retracts to $\mathcal{N}(\mathcal{I}(P))$.*

Proof. By Lemma 6, $\delta_R(\mathcal{N}(P))$ retracts to its subdigraph G induced by the vertices (X, Y) such that Y is the common outneighbourhood of all vertices in X and X is the common inneighbourhood of all vertices in Y . We will show that $G = \mathcal{N}(\mathcal{I}(P))$.

For $(X, Y) \in V(G)$, suppose that there exists a vertex u of $\mathcal{N}(P)$ not contained in X or Y . Since $u \notin X$, u is not a common inneighbor of the vertices in Y . Thus, there exists a vertex $y \in Y$ such that $u \geq y$ in P . Likewise, $u \notin Y$ so there exists a vertex $x \in X$ such that $x \geq u$. By transitivity, $x \geq y$, contradicting the fact that there is an arc from x to y in K . Therefore, $Y = \overline{X}$.

Hence the elements of G are determined by their first coordinates. It is easy to see that these first coordinates are ideals of P , that is, elements of $\mathcal{I}(P)$. Indeed, if $(X, \overline{X}) \in V(G)$ and $x < y \in X$, then (y, x) is not an arc of G , hence $x \notin \overline{X}$. So it only remains to show that adjacency in G corresponds to adjacency in $\mathcal{N}(\mathcal{I}(P))$. By definition $((X, \overline{X}), (Y, \overline{Y}))$ is an arc of G if and only if \overline{X} intersects Y , that is, $X \not\subseteq Y$, which is equivalent to $(X, Y) \in A(\mathcal{N}(\mathcal{I}(P)))$. \square

Corollary 8. *For any poset P and any integer k , $\delta_R^k(\mathcal{N}(P))$ retracts to $\mathcal{N}(\mathcal{I}^k(P))$.*

2.3. Symmetric restrictions. Let $[G]$ denote the symmetric restriction of a digraph G , that is, the subgraph spanned by its symmetric arcs.

Lemma 9. *For any poset P , $[\mathcal{N}(P)]$ retracts to a complete subgraph with cardinality equal to the width of P .*

Proof. Any antichain in P is a complete subgraph of $[\mathcal{N}(P)]$. Let $K = [A]$, where A is a maximum antichain in P . Then K is a complete subgraph of $[\mathcal{N}(P)]$ of cardinality equal to the width w of P . By Dilworth's theorem, there exists a chain partition $\{C_1, \dots, C_w\}$ of P . Each chain C_i is an independent set in $[\mathcal{N}(P)]$. Therefore, the map $\rho : [\mathcal{N}(P)] \rightarrow K$ mapping each C_i to its intersection with A is a retraction of $[\mathcal{N}(P)]$ to K . \square

To summarize the proof of Theorem 3, the existence of a n -colouring of $\delta^k(G)$ is equivalent to the existence of a homomorphism of G to $\delta_R^k(K_n)$ by Corollary 5. Since $K_n = \mathcal{N}(\overline{K}_n)$, this is equivalent to the existence of a homomorphism of G to $\mathcal{N}(\mathcal{I}^k(\overline{K}_n))$ by Corollary 8. When G is undirected, its edges must be mapped to those of $[\mathcal{N}(\mathcal{I}^k(\overline{K}_n))]$ which retracts to $K_{b(n,k)}$ by Lemma 9. This concludes the proof of Theorem 3.

3. FURTHER COMMENTS

For any poset P , the chromatic number of $\mathcal{N}(P)$ is equal to the number of elements in P , since for any two elements of P are joined by an arc in at least one direction. In particular, the number of elements of $\mathcal{I}^k(\overline{K}_n)$ is equal to the maximum possible chromatic number of a digraph G with the property that $\chi(\delta^k(G)) = n$. Note that $G = \mathcal{N}(\mathcal{I}^k(\overline{K}_n))$ achieves the bound, as well as any spanning tournament in $\mathcal{I}^k(\overline{K}_n)$. In particular, the maximum possible chromatic number of a digraph G with the property that $\chi(\delta^k(G)) = n$ is achieved with G being a transitive tournament. Specializing to $k = 2$, this implies that the n -th Dedekind number is equal to the maximum cardinality of a transitive tournament T such that $\chi(\delta^2(T)) = 2$.

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